R & D NOTES

A Correlating Equation for Combined Laminar Forced and Free Convection Heat Transfer to Power-Law Fluids

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Heat transfer by forced or free convection in laminar boundary layer flows of power law fluids has been theoretically studied by a number of investigators for large Prandtl number situations. It is well known, that, in any heat transfer circumstance, density differences are bound to arise. And, in the presence of a force field, free convection effects result. If, in a forced convection circumstance, the forces and the momentum transport rates are very large, the effects of free convection may be neglected. On the other hand, if buoyancy forces are of greater relative magnitude, forced convection effects may be ignored. But in many practical circumstances of heat transfer, the two effects may be of comparable order, and it is the combined effect that actually determines the rate of heat transfer. This note provides an equation for predicting the combined convection heat transfer rate to power law fluids, for the simplest situation of flow, past an isothermal vertical flat plate.

The only theoretical analysis for combined convection heat transfer to power law fluids past a vertical flat plate is that of Kubair and Pei (1968), which does not appear to be correct.

Their controlling parameter $Gr/Re^{2/2-n}$ can be simplified using $U_\infty = Ax^m$ and $(T_w - T_\infty) = Bx^{n'}$ to give:

$$\frac{Gr}{Re^{\frac{2}{2-n}}} = \frac{g\beta B}{A^2}\, x^{1+n'-2m} \tag{1}$$

which obviously is a constant only when $n' = 2m - 1$. This gives exactly the same restriction as imposed by Sparrow et al. (1959), when they obtained the solution for combined convection to a Newtonian fluid. However, Kubair and Pei (1968) claim that the dimensionless forms of their boundary layer equations reduce for Newtonian fluids to those proposed by Sparrow et al. (1959), except that the restriction of $n' = 2m - 1$ in their paper has been removed. This is definitely not true.

Further, a look at their generalised Prandtl number shows that it would be independent of x only when

$$m + 1 - \frac{2n}{1+n} - \frac{2m(2-n)}{1+n} = 0 \tag{2}$$

This, when solved gives the condition

$$m = \frac{1}{3} \quad \text{for} \quad n \neq 1 \tag{3}$$

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giving

$$n' = 2m - 1 = -\frac{1}{3} \tag{4}$$

This is precisely the unrealistic restriction found by Na and Hansen (1966) for similarity, to exist in the case of pure free convection to power law fluids.

Finally, the dimensionless groups of Kubair and Pei (1968) do not satisfy the continuity equation as can be seen by proper substitution and simplification.

In view of the fact that there exists no correct solution for mixed convection heat transfer to power law fluids, it is fitting to obtain at least a correlating equation to get some estimate of the relative importance of the individual forced and free convection effects in a heat transfer circumstance.

Churchill (1977) reviewed the case of laminar, assisting, forced and free convection heat transfer to Newtonian fluid and suggested the following expression to best represent the data.

$$Nu_{x,M}^3 = Nu_{x,F}^3 + Nu_{x,N}^3 \tag{5}$$

where $Nu_{x,M}$ is the Nusselt number for mixed convection, $Nu_{x,F}$ is the Nusselt number for pure forced convection, and $Nu_{x,N}$ is the Nusselt number for pure free convection.

Recently, Ruckenstein (1978) supported the choice of Churchill (1977), but provided a simple approach to interpolate the two extremes of forced and free convection, when each extreme is treated in terms of a boundary layer approximation.

In what follows, the approach of Ruckenstein (1978) is used to obtain interpolating equations for heat transfer from an isothermal vertical flat plate to power law fluids. The velocity components u and v for mixed convection can be written

$$u = u_1 + u_2 \quad \text{and} \quad v = v_1 + v_2 \tag{6}$$

where u_1, v_1 are the velocity components for pure forced convection and u_2, v_2 are the velocity components for pure free convection.

As assumption of large Prandtl numbers is now made, and the temperature field is seen to satisfy the equation

$$(u_1 + u_2)\frac{\partial T}{\partial x} + (v_1 + v_2)\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{7}$$

while the velocities u_2 and v_2 due to free convection satisfy the equations

$$0 = K \frac{\partial}{\partial y}\left(\frac{\partial u_2}{\partial y}\right)^n + g\beta(T - T_\infty) \tag{8}$$

and

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{9}$$

It is worth realizing at this point, that for power law fluids, which are known to have high consistencies, the Prandtl numbers are large, and hence, the present solution will be even more meaningful. Note also that the inertial (nonlinear) terms in the momentum Equation (8) are neglected, in comparison to the other terms. This is justifiable under the large Prandtl number assumption within the thermal boundary layer.

Using the arguments similar to those of Ruckenstein (1978), a solution is now sought as follows: Considering the region very near the solid wall, only the first term of the velocity profile provided by Acrivos et al. (1960) for pure forced convection is used to obtain the following expressions for u_1 and v_1

$$u_1 = \left\{\frac{117}{560(n+1)}\right\}^{\frac{1}{n+1}} \left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Re_x^{\frac{3}{(n+1)(2-n)}} x^{-\frac{2}{2-n}} y \tag{10}$$

$$v_1 = \frac{1}{2(n+1)}\left\{\frac{117}{560(n+1)}\right\}^{\frac{1}{n+1}} \left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Re_x^{\frac{3}{(n+1)(2-n)}} x^{-\frac{4-n}{2-n}} y^2 \tag{11}$$

Similarly, using the first term of the velocity profile provided by Shenoy and Ulbrecht (1979) for pure free convection, the following expressions for u_2 and v_2 are obtained

$$u_2 = \left(\frac{54}{1000f(n)}\right)^{\frac{1}{3n+1}} \left(\frac{3n+1}{2n+1}\right)^{\frac{1}{3n+1}} \left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Gr_x^{\frac{3}{2(n+1)(2-n)}} Pr_{x,N}^{-\frac{1}{3n+1}} x^{-\frac{2}{2-n}} y \tag{12}$$

$$v_2 = -\frac{1}{2(3n+1)}\left(\frac{54}{1000f(n)}\right)^{\frac{1}{3n+1}} \left(\frac{3n+1}{2n+1}\right)^{\frac{1}{3n+1}} \left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Gr_x^{\frac{3}{2(n+1)(2-n)}} Pr_{x,N}^{-\frac{1}{3n+1}} x^{-\frac{4-n}{2-n}} y^2 \tag{13}$$

where

$$f(n) = \frac{1}{15} - \frac{5}{126n} + \frac{1}{84n^2} - \frac{1}{486n^3} + \frac{1}{5103n^4} - \frac{1}{124740n^5} \tag{14}$$

The meanings of each of the symbols used above are given in the Notation. For $n = 1$, Equation (10)-(13) reduce to the forms used by Ruckenstein (1978), except for the values of the constants which are slightly different in each case. These constants depend on the choice of the velocity profiles made by the different authors during the derivation for the individual pure forced or pure free convection cases. But this should not be of much concern, as only the order of the terms are important during the present procedure of estimation, and these have been achieved through Equations (10)-(13).

Now, the temperature field near the solid wall satisfies Equation (7) with u_1, v_1, u_2 and v_2 given by Equations (10)-(13). Let the thickness of the thermal boundary layer δ_T be denoted by k/h_x, where k is the thermal conductivity and h_x is the local heat transfer coefficient. Evaluating each of the terms of Equation (7) by replacing ∂T by ΔT, ∂x by x, ∂y by δ_T and y by δ_T gives

$$(u_1 + u_2)\frac{\partial T}{\partial x} \sim \left[a_1\left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Re_x^{\frac{3}{(n+1)(2-n)}} x^{-\frac{2}{2-n}} \delta_T + b_1\left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Gr_x^{\frac{3}{2(n+1)(2-n)}} Pr_{x,N}^{-\frac{1}{3n+1}} x^{-\frac{2}{2-n}} \delta_T\right]\frac{\Delta T}{x} \tag{15a}$$

$$(v_1 + v_2)\frac{\partial T}{\partial y} \sim \left[a_2\left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Re_x^{\frac{3}{(n+1)(2-n)}} x^{-\frac{4-n}{2-n}} \delta_T^2 + b_2\left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Gr_x^{\frac{3}{2(n+1)(2-n)}} Pr_{x,N}^{-\frac{1}{3n+1}} x^{-\frac{4-n}{2-n}} \delta_T^2\right]\frac{\Delta T}{\delta_T} \tag{15b}$$

and

$$\frac{\delta^2 T}{\partial y^2} \sim \frac{\Delta T}{\delta_T^2} \tag{15c}$$

In the above a_1, b_1, a_2 and b_2 are functions of n alone and the symbol $\sim$ stands for 'the order of'.

Replacing each of the terms of Equation (7) with the corresponding expression in (15) gives

$$a\left(\frac{k}{\rho}\right)^{\frac{1}{2-n}} Re_x^{\frac{3}{(n+1)(2-n)}} x^{-\frac{4-n}{2-n}} \delta_T + b\left(\frac{K}{\rho}\right)^{\frac{1}{2-n}} Gr_x^{\frac{3}{2(n+1)(2-n)}} Pr_{x,N}^{-\frac{1}{3n-1}} x^{-\frac{4-n}{2-n}} \delta_T = \frac{\alpha}{\delta_T^2} \tag{16}$$

where a and b are functions of n alone and will be determined later. Now as

$$Nu_{x,M} \equiv \frac{x}{\delta_T} \tag{17}$$

Equation (16) can be rewritten after simplification as

$$Nu_{x,M}^3 = a\, Re_x^{\frac{3}{n+1}} Pr_{x,F} + b\, Gr_x^{\frac{3}{2(n+1)}} Pr_{x,N}^{\frac{3n}{3n+1}} \tag{18}$$

where

$$Re_x = \frac{\rho U_\infty^{2-n} x^n}{K} \tag{19a}$$

$$Pr_{x,F} = \frac{1}{\alpha}\left(\frac{K}{\rho}\right)^{\frac{2}{n+1}} x^{\frac{1-n}{n+1}} U_\infty^{\frac{3(n-1)}{n+1}} \tag{19b}$$

$$Gr_x = \frac{\rho^2 x^{n+2}[g\beta(T_w - T_\infty)]^{2-n}}{K^2} \tag{19c}$$

$$Pr_{x,N} = \frac{1}{\alpha}\left(\frac{K}{\rho}\right)^{\frac{2}{n+1}} x^{\frac{n-1}{2(n+1)}} [g\beta(T_w - T_\infty)]^{\frac{3(n-1)}{2(n+1)}} \tag{19d}$$

The coefficients a and b can be obtained by making use of the fact that for an isothermal vertical flat plate under large Prandtl number situations

$$Nu_{x,F} = \frac{1}{0.893}\left[\frac{1}{18}\left\{\frac{117}{560(n+1)}\right\}^{\frac{1}{n+1}}\right]^{\frac{1}{3}} \left[\frac{2n+1}{n+1}\right]^{\frac{1}{3}} Re_x^{\frac{1}{n+1}} Pr_{x,F}^{\frac{1}{3}} \tag{20}$$

for pure forced convection heat transfer to power law fluids as given by Acrivos et al. (1960) and

$$Nu_{x,N} = 2\left[\frac{1}{2}\left(\frac{3}{10}\right)^{\frac{1}{n}} f(n)\right]^{\frac{n}{3n+1}} \left[\frac{2n+1}{3n+1}\right]^{\frac{n}{3n+1}} Gr_x^{\frac{1}{2(n+1)}} Pr_{x,N}^{\frac{n}{3n+1}} \quad (21)$$

for pure free convection heat transfer to power law fluids as given by Shenoy and Ulbrecht (1979).

Thus, it can be easily seen that the correlating equation for mixed convection flow of Newtonian fluids suggested by Churchill (1977) and later established by Ruckenstein (1978), using the approximate interpolation procedure, holds good even for non-Newtonian power law fluids.

The resulting equation is again

$$Nu_{x,M}^3 = Nu_{x,F}^3 + Nu_{x,N}^3 \quad (22)$$

except for the new definitions of $Nu_{x,F}$ and $Nu_{x,N}$ as given by Equation (20) and (21) respectively.

Finally, it is worth mentioning that, as an assumption of large Prandtl numbers was necessary to obtain the present solution, it would be natural to expect the results from the above equations to be more correct for non-Newtonian fluids (which generally have higher consistencies, implying larger Prandtl numbers) then Newtonian fluids. It is unfortunate that due to lack of experimental data in the literature on combined laminar forced and free convection heat transfer to power law fluids past an isothermal vertical flat plate, a comparison cannot be made to validate the propriety of the above equations.

NOTATION

a = function of n in Equation (16)
a_1 = function of n in Equation (15*a*)
a_2 = function of n in Equation (15*b*)
A = constant in Equation (1)
u_1 = velocity component for forced convection in the direction along the plate
u_2 = velocity component for free convection in the direction along the plate
U_∞ = free stream velocity for forced convection
v = velocity component for mixed convection in the direction normal to the plate
v_1 = velocity component for forced convection in the direction normal to the plate
v_2 = velocity component for free convection in the direction normal to the plate
x = distance along the plate from the leading edge
y = distance normal to the plate

Greek Letters

α = thermal diffusivity of the fluid
β = expansion coefficient of the fluid
δ_T = thermal boundary layer thickness
ρ = density of the fluid

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Stability of the High Yield MSMPR Crystallizer with Size-Dependent Growth Rate

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Cycling behavior of well-stirred isothermal crystallizers occurs in both laboratory-scale and commercial plants, and has been studied as summarized by Randolph and Larson (1971 Chap. 5) and in a recent review paper (Randolph 1977).

The high yield, mixed suspension, mixed product removal (MSMPR) crystallizer has the well-known stability criterion $d(\log B^0)/d(\log G) < 21$. This idealized crystallizer serves as a reference case to judge stability in other configurations and/or systems. Sherwin, Shinnar and Katz (1969), and Randolph, Beer and Keener (1973) investigated the stability limits of mixed suspension classified product removal (MSCPR) crystallizers. They found the stable region is significantly reduced, compared with the MSMPR case. Size-dependent growth rate affects CSD in a manner qualitatively the same as size-dependent product removal; both result in varying slopes on a semi-log population density plot, depending on growth and/or removal rate at that size. Thus the question naturally arises, "Does size-dependent growth rate also destabilize CSD?"

The limiting case of such size-dependent growth rates has been studied. Sherwin, Shinnar and Katz (1967) predicted that increasing growth rate with increasing size raises the system stability limits. Their model assumed that growth rate was linearly proportional to the crystal size. Anshus and Ruckenstein (1973) observed that the CSD stability region narrows when the growth rate is mass transfer controlling, that is, size-dependent and inversely proportional to a power of the size (below the

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